HOMOLOGICAL PROPERTIES OF BIGRADED ALGEBRAS

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Abstract. We investigate the x- and y-regularity of a bigraded K-algebra R as introduced in [2]. These notions are used to study asymptotic properties of certain finitely generated bigraded modules. As an application we get for any equigenerated graded ideal I upper bounds for the number j_0 for which $\operatorname{reg}(I^j)$ is a linear function for $j \geq j_0$. Finally, we give upper bounds for the x- and y-regularity of generalized Veronese algebras.

Introduction

Let $S = K[x_1, \dots , x_n, y_1, \dots , y_m]$ be a standard bigraded polynomial ring with $\deg(x_i) = (1, 0)$ and $\deg(y_j) = (0, 1)$, and let $J \subset S$ be a bigraded ideal. In this paper we study homological properties of the bigraded algebra $R = S/J$.

First we consider the x- and the y-regularity of R. According to [2] they are defined as follows:

$$\operatorname{reg}^S_x(R) = \max\{a \in \mathbb{Z}\colon \beta^S_{i,(a+i,b)}(R) \neq 0 \text{ for some } i, b \in \mathbb{Z}\},$$

$$\operatorname{reg}^S_y(R) = \max\{b \in \mathbb{Z}\colon \beta^S_{i,(a,b+i)}(R) \neq 0 \text{ for some } i, a \in \mathbb{Z}\}$$

where $\beta^S_{i,(a,b)}(R) = \dim_K \operatorname{Tor}^S_i(K, R)_{(a,b)}$ is the i^{th} bigraded Betti number of R in bidegree (a, b). We give a homological characterization of these regularities similarly as in the graded case (see [3]). As an application we generalize a result of Trung [13] concerning d-sequences. Furthermore we prove that

$$\operatorname{reg}^S_x(S/J) = \operatorname{reg}^S_x(S/\operatorname{bigin}(J))$$

where $\operatorname{bigin}(J)$ is the bigeneric initial ideal of J with respect to the bigraded reverse lexicographic order induced by $y_1 > \ldots > y_m > x_1 > \ldots > x_n$.

It was shown in [7] (or [12]) that for $j \gg 0$, $\operatorname{reg}(I^j)$ is a linear function $cj + d$ in j for a graded ideal I in the polynomial ring. In [12] the constant c is described in terms of invariants of I. In this paper we give, in case I is equigenerated, bounds j_0 such that for $j \geq j_0$ the function is linear and give also a bound for d. Our methods can also be applied to $\operatorname{reg}(S^j(I))$, where $S^j(I)$ is the j^{th} symmetric power of I.

In the last section we introduce a generalized Veronese algebra in the bigraded setting. For a bigraded algebra R and $\tilde{\Delta} = (s, t) \in \mathbb{N}^2$ with $(s, t) \neq (0, 0)$ we set

$$R_{\tilde{\Delta}} = \bigoplus_{(a,b)\in\mathbb{N}^2} R_{(as,bt)}.$$

In the same manner as it is done for diagonal subalgebras in [6], we prove that for these algebras

$$\operatorname{reg}_x^{S_{\tilde\Delta}}(R_{\tilde\Delta}) = 0 \text{ and } \operatorname{reg}_y^{S_{\tilde\Delta}}(R_{\tilde\Delta}) = 0, \text{ if } s \gg 0 \text{ and } t \gg 0.$$

1. Preliminaries

Throughout this paper, let K be an infinite field and $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$ be a standard bigraded polynomial ring with $\deg(x_i) = (1,0)$ and $\deg(y_j) = (0,1)$. Let M be a finitely generated bigraded S-module. For some bihomogeneous $w \in M$ with $\deg(w) = (a,b)$ we set $\deg_x(w) = a$ and $\deg_y(w) = b$. Sometimes we will consider the $\mathbb{Z}$-graded modules $M_{(a,*)} = \bigoplus_{b\in\mathbb{Z}} M_{(a,b)}$ or $M_{(*,b)} = \bigoplus_{a\in\mathbb{Z}} M_{(a,b)}$. If in addition M is $\mathbb{N}^n \times \mathbb{N}^m$-graded, we write $M_{(u,v)}$ for the homogeneous component in bidegree (u,v) where $u \in \mathbb{N}^n$ and $v \in \mathbb{N}^m$. For $u \in \mathbb{N}^n$ we set $\operatorname{supp}(u) = \{i \colon u_i > 0\}$.

Define $\mathfrak{m}_x = (x_1, \dots, x_n) = (\mathbf{x})$, $\mathfrak{m}_y = (y_1, \dots, y_m) = (\mathbf{y})$ and $\mathfrak{m} = \mathfrak{m}_x + \mathfrak{m}_y$. Let $S_x = K[x_1, \dots, x_n]$ and $S_y = K[y_1, \dots, y_m]$ be the polynomial rings with respect to the x-variables and the y-variables.

For some $u = (u_1, \dots, u_n) \in \mathbb{N}^n$ and $v = (v_1, \dots, v_m) \in \mathbb{N}^m$ we write $x^u y^v$ for the monomial $x_1^{u_1} \dots x_n^{u_n} y_1^{v_1} \dots y_m^{v_m}$. For $u, u' \in \mathbb{N}^n$ let $u \preceq u'$, if $u_i \leq u'_i$ for all i. Furthermore we set $|u| = u_1 + \ldots + u_n$. Let $\varepsilon_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^n$ where the entry 1 is at the i^{th} position. For $t \in \mathbb{N}$ define $[t] = \{1, \dots, t\}$.

We consider bigraded algebras $R = S/J$, which are quotients of S by some bigraded ideal J. For a finitely generated bigraded R-module M and $a, b \in \mathbb{N}$ let $\beta^R_{i,(a,b)}(M) = \dim_K \operatorname{Tor}_i^R(M,K)_{(a,b)}$ be the i^{th} bigraded Betti number in bidegree (a,b). We recall from [2] that

$$\operatorname{reg}_x^R(M) = \sup\{a \in \mathbb{Z} \colon \beta^R_{i,(a+i,b)}(M) \neq 0 \text{ for some } i, b \in \mathbb{Z}\},$$

$$\operatorname{reg}_y^R(M) = \sup\{b \in \mathbb{Z} \colon \beta^R_{i,(a,b+i)}(M) \neq 0 \text{ for some } i, a \in \mathbb{Z}\}$$

is the x- and y-regularity of M. For $R = S$ we set $\operatorname{reg}_x(M) = \operatorname{reg}_x^S(M)$ and $\operatorname{reg}_y(M) = \operatorname{reg}_y^S(M)$.

Let $K_\bullet(k,l;M)$ denote the Koszul complex of M and $H_\bullet(k,l;M)$ the Koszul homology of M with respect to $x_1, \dots, x_k$ and $y_1, \dots, y_l$ (see [5] for details). If it is clear from the context, we write $K_\bullet(k,l)$ and $H_\bullet(k,l)$ instead of $K_\bullet(k,l;M)$ and $H_\bullet(k,l;M)$. Note that $K_\bullet(k,l;M) = K_\bullet(k,l;S) \otimes_S M$ where $K_\bullet(k,l;S)$ is the exterior algebra on $e_1, \dots, e_k$ and $f_1, \dots, f_l$ with $\deg(e_i) = (1,0)$ and $\deg(f_j) = (0,1)$ together with a differential ∂ induced by $\partial(e_i) = x_i$ and $\partial(f_j) = y_j$. For a cycle $z \in K_\bullet(k,l;M)$ we denote with $[z] \in H_\bullet(k,l;M)$ the corresponding homology class. There are two long exact sequences relating the homology groups:

$$\dots \to H_i(k,l;M)(-1,0) \overset{x_{k+1}}{\to} H_i(k,l;M) \to H_i(k+1,l;M) \to H_{i-1}(k,l;M)(-1,0)$$
$$\overset{x_{k+1}}{\to} \dots \to H_0(k,l;M)(-1,0) \overset{x_{k+1}}{\to} H_0(k,l;M) \to H_0(k+1,l;M) \to 0$$

and

$$\dots \to H_i(k,l;M)(0,-1) \overset{y_{l+1}}{\to} H_i(k,l;M) \to H_i(k,l+1;M) \to H_{i-1}(k,l;M)(0,-1)$$
$$\overset{y_{l+1}}{\to} \dots \to H_0(k,l;M)(0,-1) \overset{y_{l+1}}{\to} H_0(k,l;M) \to H_0(k,l+1;M) \to 0.$$

The map $H_i(k,l;M) \to H_i(k+1,l;M)$ is induced by the inclusion of the corresponding Koszul complexes. Every homogeneous element $z \in K_\bullet(k+1,l;M)$ can be uniquely written as $e_{k+1} \wedge z' + z''$ with $z', z'' \in K_\bullet(k,l;M)$. Then $H_i(k+1,l;M) \to H_{i-1}(k,l;M)(-1,0)$ is given by sending $[z]$ to $[z']$. Furthermore $H_i(k,l;M)(-1,0) \overset{x_{k+1}}{\to} H_i(k,l;M)$ is just the multiplication with x_{k+1}. The maps in the other exact sequence are analogue.

2. Regularity

Let R be a bigraded algebra. To simplify the notation we do not distinguish between the polynomial ring variables x_i or y_j and the corresponding residue classes in R. Following [3] (or [13] under the name filter regular element) we call an element $x \in R_{(1,0)}$ *an almost regular element for* R (with respect to the x-degree) if

$$(0 :_R x)_{(a,*)} = 0 \text{ for } a \gg 0.$$

A sequence $x_1, \dots, x_t \in R_{(1,0)}$ is *an almost regular sequence* (with respect to the x-degree) if for all $i \in [t]$ the x_i is almost regular for $R/(x_1, \dots, x_{i-1})R$.

Analogue we call an element $y \in R_{(0,1)}$ *an almost regular element for* R (with respect to the y-degree) if

$$(0 :_R y)_{(*,b)} = 0 \text{ for } b \gg 0.$$

A sequence $y_1, \dots, y_t \in R_{(0,1)}$ is *an almost regular sequence* (with respect to the y-degree) if for all $i \in [t]$ the y_i is almost regular for $R/(y_1, \dots, y_{i-1})R$.

It is well-known that, provided $|K| = \infty$, after a generic choice of coordinates we can achieve that a K-basis of $R_{(1,0)}$ is almost regular for R with respect to the x-degree and a K-basis of $R_{(0,1)}$ is almost regular for R with respect to the y-degree. For the convenience of the reader we give a proof of this fact, which follows from the following lemma (see also [13]).

Lemma 2.1. *Let R be a bigraded algebra. If $\dim_K R_{(1,0)} > 0$ ($\dim_K R_{(0,1)} > 0$), then there exists an element $x \in R_{(1,0)}$ ($y \in R_{(0,1)}$) which is almost regular for R.*

Proof. By symmetry it is enough to prove the existence of x. We claim that it is possible to choose $0 \neq x \in R_{(1,0)}$ such that for all $Q \in \operatorname{Ass}_S(0 :_R x)$ one has $Q \supseteq \mathfrak{m}_x$. It follows that $\operatorname{Rad}_S(\operatorname{Ann}_S(0 :_R x)) \supseteq \mathfrak{m}_x$. Hence there exists an integer i such that $\mathfrak{m}_x^i(0 :_R x) = 0$ and this proves the lemma.

It remains to show the claim. If $P \supseteq \mathfrak{m}_x$ for all $P \in \operatorname{Ass}_S(R)$, then we may choose $0 \neq x \in R_{(1,0)}$ arbitrary because $\operatorname{Ass}_S(0 :_R x) \subseteq \operatorname{Ass}_S(R)$. Otherwise there exists an ideal $P \in \operatorname{Ass}_S(R)$ with $P \not\supseteq \mathfrak{m}_x$. In this case we may choose $x \in R_{(1,0)}$ such that

$$x \notin \bigcup_{P \in \operatorname{Ass}_S(R), P \not\supseteq \mathfrak{m}_x} P$$

since $|K| = \infty$. Let $Q \in \operatorname{Ass}_S(0 :_R x)$ be arbitrary. Then $x \in Q$ because $x \in \operatorname{Ann}_S(0 :_R x)$. We also have that $Q \in \operatorname{Ass}_S(R)$ and this implies that $Q \supseteq \mathfrak{m}_x$ by the choice of x. This gives the claim. $\square$

Let $\mathbf{x}$ and $\mathbf{y}$ be almost regular for R with respect to the x- and y-degree. Define

$$s_i^x = \max(\{a\colon (0 :_{R/(x_1,\dots,x_{i-1})R} x_i)_{(a,*)} \neq 0\} \cup \{0\}), \quad s^x = \max\{s_1^x,\dots,s_n^x\}$$

and

$$s_i^y = \max(\{b\colon (0 :_{R/(y_1,\dots,y_{i-1})R} y_i)_{(*,b)} \neq 0\} \cup \{0\}), \quad s^y = \max\{s_1^y,\dots,s_m^y\}.$$

The following theorem characterizes the x- and y-regularity. It is the analogue of the corresponding graded version in [3].

For its proof we consider $\tilde{H}_0(k-1,0) = (0 :_{R/(x_1,\dots,x_{k-1})R} x_k)$ for $k \in [n]$ and $\tilde{H}_0(n,k-1) = (0 :_{R/(\mathfrak{m}_x+y_1,\dots,y_{k-1})R} y_k)$ for $k \in [m]$. Then the beginning of the long exact Koszul sequence of the Koszul homology groups of R for $k \in [n]$ is modified to

$$\dots \to H_1(k-1,0)(-1,0) \xrightarrow{x_k} H_1(k-1,0) \to H_1(k,0) \to \tilde{H}_0(k-1,0)(-1,0) \to 0,$$

and for $k \in [m]$ to

$$\dots \to H_1(n,k-1)(0,-1) \xrightarrow{y_k} H_1(n,k-1) \to H_1(n,k) \to \tilde{H}_0(n,k-1)(0,-1) \to 0.$$

Note that for $k \in [n]$ and $i \geq 1$ one has $H_i(k,0)_{(a,*)} = 0$ for $a \gg 0$. Similarly for $k \in [m]$ and $i \geq 1$ one has $H_i(n,k)_{(*,b)} = 0$ for $b \gg 0$.

Theorem 2.2. *Let R be a bigraded algebra, $\mathbf{x}$ almost regular for R with respect to the x-degree and $\mathbf{y}$ almost regular for R with respect to the y-degree. Then*

$$\operatorname{reg}_x(R) = s^x \text{ and } \operatorname{reg}_y(R) = s^y.$$

Proof. By symmetry it is enough to show this theorem only for $\mathbf{x}$. Let

$$r_{(k,0)} = \max(\{a\colon H_i(k,0)_{(a+i,*)} \neq 0 \text{ for } i \in [k]\} \cup \{0\})$$

for $k \in [n]$ and

$$r_{(n,k)} = \max(\{a\colon H_i(n,k)_{(a+i,*)} \neq 0 \text{ for } i \in [n+k]\} \cup \{0\})$$

for $k \in [m]$. Then $r_{(n,m)} = \operatorname{reg}_x(R)$ because $H_0(n,m) = K$. We claim that:

(i) For $k \in [n]$ one has $r_{(k,0)} = \max\{s_1^x,\dots,s_k^x\}$.
(ii) For $k \in [m]$ one has $r_{(n,k)} = \max\{s_1^x,\dots,s_n^x\}$.

This yields the theorem. We show (i) by induction on $k \in [n]$. For $k = 1$ we have the following exact sequence

$$0 \to H_1(1,0) \to \tilde{H}_0(0,0)(-1,0) \to 0$$

which proves this case. Let $k > 1$. Since

$$\dots \to H_1(k,0) \to \tilde{H}_0(k-1,0)(-1,0) \to 0,$$

we get $r_{(k,0)} \geq s_k^x$. If $r_{(k-1,0)} = 0$, then $r_{(k,0)} \geq r_{(k-1,0)}$. Assume that $r_{(k-1,0)} > 0$. There exists an integer i such that $H_i(k-1)_{(r_{(k-1,0)}+i,*)} \neq 0$. Then by

$$\dots \to H_{i+1}(k,0)_{(r_{(k-1,0)}+i+1,*)} \to H_i(k-1,0)_{(r_{(k-1,0)}+i,*)}$$
$$\to H_i(k-1,0)_{(r_{(k-1,0)}+i+1,*)} \to \dots$$

we have $H_{i+1}(k,0)_{(r_{(k-1,0)}+i+1,*)} \neq 0$ because $H_i(k-1,0)_{(r_{(k-1,0)}+i+1,*)} = 0$. This gives also $r_{(k,0)} \geq r_{(k-1,0)}$. On the other hand let $a > \max\{r_{(k-1,0)}, s^x_k\}$. If $i \geq 2$, then by

$$\ldots \to H_i(k-1,0)_{(a+i,*)} \to H_i(k,0)_{(a+i,*)} \to H_{i-1}(k-1,0)_{(a+i-1,*)} \to \ldots$$

we get $H_i(k,0)_{(a+i,*)} = 0$ because $H_i(k-1,0)_{(a+i,*)} = H_{i-1}(k-1,0)_{(a+i-1,*)} = 0$. Similarly $H_1(k,0)_{(a+1,*)} = 0$. Therefore we obtain that $r_{(k,0)} = \max\{r_{(k-1,0)}, s^x_k\} = \max\{s^x_1, \ldots, s^x_k\}$ by the induction hypothesis.

We prove (ii) also by induction on $k \in \{0, \ldots, m\}$. The case $k = 0$ was shown in (i), so let $k > 0$. Assume that $a > s^x$. For $i \geq 2$ one has

$$\ldots \to H_i(n,k-1)_{(a+i,*)} \to H_i(n,k)_{(a+i,*)} \to H_{i-1}(n,k-1)_{(a+i,*)} \to \ldots .$$

Then we get $H_i(n,k)_{(a+i,*)} = 0$ because $H_i(n,k-1)_{(a+i,*)} = H_{i-1}(n,k-1)_{(a+i,*)} = 0$. Similarly $H_1(n,k)_{(a+1,*)} = 0$ and therefore $r_{(n,k)} \leq s^x$. If $s^x = 0$, then $r_{(n,k)} = s^x$. Assume that $0 < s^x = r_{(n,k-1)}$. There exists an integer i such that $H_i(n,k-1)_{(s^x+i,*)} \neq 0$. Consider

$$\ldots \to H_i(n,k-1)_{(s^x+i,*)} \xrightarrow{y_k} H_i(n,k-1)_{(s^x+i,*)} \to H_i(n,k)_{(s^x+i,*)} \to \ldots .$$

If $H_i(n,k)_{(s^x+i,*)} = 0$, then $H_i(n,k-1)_{(s^x+i,*)} = y_k H_i(n,k-1)_{(s^x+i,*)}$. This is a contradiction by Nakayamas lemma because $H_i(n,k-1)_{(s^x+i,*)}$ is a finitely generated S_y-module. Hence $H_i(n,k)_{(s^x+i,*)} \neq 0$ and thus $r_{(n,k)} = s^x$. $\square$

3. d-sequences and s-sequences

The concept of a d-sequence was introduced by Huneke [11]. Recall that a sequence of elements $f_1, \ldots, f_r$ in a ring is called a *d-sequence*, if

(i) $f_1, \ldots, f_r$ is a minimal system of generators of the ideal $I = (f_1, \ldots, f_r)$.
(ii) $(f_1, \ldots, f_{i-1}) : f_i \cap I = (f_1, \ldots, f_{i-1})$.

A result in [13] motivated the following theorem. For a bigraded algebra R let n_x denote the ideal generated by the $(1,0)$-forms of R and let n_y denote the ideal generated by the $(0,1)$-forms of R.

Proposition 3.1. *Let R be a bigraded algebra. Then:*

(i) $\mathrm{reg}_x(R) = 0$ *if and only if a generic minimal system of generators of* $(1,0)$-*forms for* n_x *is a d-sequence.*
(ii) $\mathrm{reg}_y(R) = 0$ *if and only if a generic minimal system of generators of* $(0,1)$-*forms for* n_y *is a d-sequence.*

Proof. By symmetry we only have to prove (i). Without loss of generality $\mathbf{x} = x_1, \ldots, x_n$ is an almost regular sequence for R with respect to the x-degree because a generic minimal system of generators of $(1,0)$-forms for n_x has this property.

By 2.2 one has $\mathrm{reg}_x(R) = 0$ if and only if $s^x = 0$. By definition of s^x this is equivalent to the fact that, for all $i \in [n]$ and all $a > 0$, we have

$$\Big(\frac{(x_1, \ldots, x_{i-1}) :_R x_i}{(x_1, \ldots, x_{i-1})}\Big)_{(a,*)} = 0.$$

Equivalently, for all $i \in [n]$ we obtain $(x_1, \ldots, x_{i-1}) :_R x_i \cap n_x = (x_1, \ldots, x_{i-1})$. This concludes the proof. $\square$

If n_x (resp. n_y) can be generated by a d-sequence (not necessarily generic), then the proof of 3.1 shows that $\operatorname{reg}_x(R) = 0$ (resp. $\operatorname{reg}_y(R) = 0$).

For an application we recall some more definitions. Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree d. Let $R(I)$ denote the Rees algebra of I and let $S(I)$ denote the symmetric algebra of I. It is well known that both algebras are bigraded and have a presentation S/J for a bigraded ideal $J \subset S$. For example $R(I) = S_x[It] \subset S_x[t]$. Define

$$\varphi : S \to R(I),\ x_i \mapsto x_i,\ y_j \mapsto f_j t,$$

and let $J = \operatorname{Ker}(\varphi)$. With the assumption that I is generated in one degree we have that J is a bigraded ideal. Then we will always assume that $R(I) = S/J$. Note that then $I^j \cong (S/J)_{(*,j)}(-jd)$ for all $j \in \mathbb{N}$. Similarly we may assume that $S(I) = S/J$ for a bigraded ideal $J \subset S$. We also consider the finitely generated S_x-module $S^j(I) = (S/J)_{(*,j)}(-jd)$, which we call the j^{th} symmetric power of I.

For the notion of an s-sequence see [10]. The following results were shown in [10] and [13].

Corollary 3.2. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree d. Then:*

(i) *I can be generated by an s-sequence (with respect to the reverse lexicographic order) if and only if $\operatorname{reg}_y(S(I)) = 0$.*
(ii) *I can be generated by a d-sequence if and only if $\operatorname{reg}_y(R(I)) = 0$.*

Proof. In [10] and [13] it was shown that

(i) I can be generated by an s-sequence (with respect to the reverse lexicographic order) if and only if $n_y \subseteq S(I)$ can be generated by a d-sequence.
(ii) I can be generated by a d-sequence if and only if $n_y \subseteq R(I)$ can be generated by a d-sequence.

Together with 3.1 these facts conclude the proof. □

4. Bigeneric initial ideals

We recall the following definitions from [2]. For a monomial $x^u y^v \in S$ we set

$$m_x(x^u y^v) = m(u) = \max\{0, i \text{ with } u_i > 0\},$$

$$m_y(x^u y^v) = m(v) = \max\{0, i \text{ with } v_i > 0\}.$$

Similarly we define for $L \subseteq [n]$,

$$m(L) = \max\{0, i \text{ with } i \in L\}.$$

Let $J \subset S$ be a monomial ideal. Let $G(J)$ denote the unique minimal system of generators of J. If $G(J) = \{z_1, \dots, z_t\}$ with $\deg(z_i) = (u^i, v^i) \in \mathbb{N}^n \times \mathbb{N}^m$, then we set $m_x(J) = \max\{|u^i|\}$ and $m_y(J) = \max\{|v^i|\}$.

J is called *bistable* if for all monomials $z \in J$, all $i \le m_x(z)$, all $j \le m_y(z)$ one has $x_i z / x_{m_x(z)} \in J$ and $y_j z / y_{m_y(z)} \in J$. J is called *strongly bistable* if for all monomials $z \in J$, all $i \le s$ with x_s divides z, all $j \le t$ with y_t divides z one has $x_i z / x_s \in J$ and $y_j z / y_t \in J$.

Lemma 4.1. *Let $J \subset S$ be a bistable ideal and $R = S/J$. Then:*

(i) *$x_n, \dots, x_1$ is an almost regular sequence for R with respect to the x-degree.*
(ii) *$y_m, \dots, y_1$ is an almost regular sequence for R with respect to the y-degree.*

Proof. This follows easily from the fact that J is bistable. $\square$

We fix a term order $>$ on S by defining $x^u y^v > x^{u'} y^{v'}$ if either $(|u|+|v|, |v|, |u|) > (|u'|+|v'|, |v'|, |u'|)$ lexicographically or $(|u|+|v|, |v|, |u|) = (|u'|+|v'|, |v'|, |u'|)$ and $x^u y^v > x^{u'} y^{v'}$ reverse lexicographically induced by $y_1 > \dots > y_m > x_1 > \dots > x_n$ (see [8] for details on monomial orders). For a bigraded ideal J let $\mathrm{in}(J)$ denote the monomial ideal generated by $\mathrm{in}(f)$ for all $f \in J$. In [2] the bigeneric initial ideal $\mathrm{bigin}(J)$ was constructed in the following way: For $t \in \mathbb{N}$ let $\mathrm{GL}(t, K)$ be the general linear group of the $t \times t$-matrices with entries in K. Let $G = \mathrm{GL}(n, K) \times \mathrm{GL}(m, K)$ and $g = (d_{ij}, e_{kl}) \in G$. Then g defines an S automorphism by extending $g(x_j) = \sum_i d_{ij} x_i$ and $g(y_l) = \sum_k e_{kl} y_k$. There exists a non-empty Zariski open set $U \subset G$ such that for all $g \in U$ we have $\mathrm{bigin}(J) = \mathrm{in}(gJ)$. We call these $g \in U$ *generic* for J. If $\mathrm{char}(K) = 0$, then $\mathrm{bigin}(J)$ is strongly bistable for every bigraded ideal J. See for example [3] for similar results in the graded case.

Proposition 4.2. *Let $J \subset S$ be a bigraded ideal. If $\mathrm{char}(K) = 0$, then*

$$\mathrm{reg}_x(S/J) = \mathrm{reg}_x(S/\mathrm{bigin}(J)).$$

Proof. Set $\mathbf{x} = x_n, \dots, x_1$, choose $g \in G$ generic for J and let $\tilde{\mathbf{x}} = \tilde{x}_n, \dots, \tilde{x}_1$ such that $x_i = g(\tilde{x}_i)$. We may assume that the sequence $\tilde{\mathbf{x}}$ is almost regular for S/J with respect to the x-degree. Furthermore by 4.1 the sequence $\mathbf{x}$ is almost regular for $S/\mathrm{bigin}(J)$ with respect to the x-degree. We have

$$(0 :_{S/((\tilde{x}_n, \dots, \tilde{x}_{i+1}) + J)} \tilde{x}_i) \cong (0 :_{S/((x_n, \dots, x_{i+1}) + g(J))} x_i).$$

It follows from [8, 15.12] that

$$(0 :_{S/((x_n, \dots, x_{i+1}) + g(J))} x_i) \cong (0 :_{S/((x_n, \dots, x_{i+1}) + \mathrm{bigin}(J))} x_i).$$

By 2.2 we get the desired result. $\square$

Remark 4.3. (i) In general it is not true that

$$\mathrm{reg}_y(S/J) = \mathrm{reg}_y(S/\mathrm{bigin}(J)).$$

For example let $S = K[x_1, \dots, x_3, y_1, \dots, y_3]$ and $J = (y_2 x_2 - y_1 x_3, y_3 x_1 - y_1 x_3)$. Then the minimal bigraded free resolution of S/J is given by

$$0 \to S(-2,-2) \to S(-1,-1) \oplus S(-1,-1) \to S \to 0.$$

Therefore $\mathrm{reg}_x(S/J) = 0$ and $\mathrm{reg}_y(S/J) = 0$. On the other hand $\mathrm{bigin}(J) = (y_2 x_1, y_1 x_1, y_1^2 x_2)$ with the minimal bigraded free resolution of $S/\mathrm{bigin}(J)$

$$0 \to S(-2,-2) \oplus S(-1,-2)$$
$$\to S(-1,-1) \oplus S(-1,-1) \oplus S(-1,-2) \to S \to 0.$$

Hence $\mathrm{reg}_x(S/\mathrm{bigin}(J)) = 0$ and $\mathrm{reg}_y(S/\mathrm{bigin}(J)) = 1$.

(ii) It is easy to calculate the x- and the y-regularity of bistable ideals. In fact, in [2] it was shown that for a bistable ideal $J \subset S$ we have

$$\operatorname{reg}_x(J) = m_x(J) \text{ and } \operatorname{reg}_y(J) = m_y(J).$$

5. Regularity of powers and symmetric powers of ideals

Consider a bigraded algebra $R = S/J$ where J is a bistable ideal. Note that by 4.1 the sequence $x_n, \dots, x_1$ is almost regular for R with respect to the x-degree. For $i \in [n]$ and $j \geq 0$ we define

$$m_j^i = \max(\{a \in \mathbb{N} \colon (0 :_{R/(x_n,\dots,x_{i+1})R} x_i)_{(a,j)} \neq 0\} \cup \{0\}).$$

Furthermore for a bistable ideal J and $v \in \mathbb{N}^n$ we set $J_{(*,v)} = I_v y^v$ where $I_v \subset S_x$ is again a monomial ideal, which is stable in the usual sense, that is if $x^u \in I_v$, then $x_i x^u / x_{m(u)} \in I_v$ for $i \leq m(u)$.

Proposition 5.1. *Let $J \subset S$ be a bistable ideal and $R = S/J$. Then:*

(i) *For every $i \in [n]$ and for $j \geq 0$ we have $m_j^i \leq \max\{m_x(J) - 1, 0\}$.*
(ii) *For every $i \in [n]$ and for $j \geq m_y(J)$ we have $m_j^i = m_{m_y(J)}^i$.*

Proof. If $G(J) = \{x^{u^k} y^{v^k} \colon k = 1, \dots, r\}$, then $I_v = (x^{u^k} \colon v^k \preceq v)$ for $v \in \mathbb{N}^n$. This means that for all $x^u \in G(I_v)$ one has $|u| \leq m_x(J)$. For fixed v with $|v| = j$ we have

$$(0 :_{R/(x_n,\dots,x_{i+1})R} x_i)_{(*,v)} = \frac{((x_n, \dots, x_{i+1}) + I_v :_{S_x} x_i)}{(x_n, \dots, x_{i+1}) + I_v} y^v.$$

As a K-vector space

$$\frac{((x_n, \dots, x_{i+1}) + I_v :_{S_x} x_i)}{(x_n, \dots, x_{i+1}) + I_v} y^v = \bigoplus_{x^u \in G(I_v), m(u) = i} K(x^u / x_{m(u)}) y^v$$

because I_v is stable. Thus $m_j^i \leq \max\{m_x(J) - 1, 0\}$, which is (i).

To prove (ii) we replace J by $J_{(*, \geq m_y(J))}$ and may assume that J is generated in y-degree $t = m_y(J)$. Then $G(J) = \{x^{u^k} y^{v^k} \colon k = 1, \dots, r\}$ where $|v^k| = t$ for all $k \in [r]$. Let $|u^k|$ be maximal with $m(u^k) = i$ and define $c^i = \max\{|u^k| - 1, 0\}$. We show that $m_j^i = c^i$ for $j \geq t$ and this gives (ii). By a similar argument as in (i) we have $m_{s+t}^i \leq c^i$ for all $s \geq 0$. If $c^i = 0$, then $m_{s+t}^i = 0$. Assume that $c^i \neq 0$. We claim that

$$(*) \quad 0 \neq [(x^{u^k}/x_i) y^{v^k} y_n^s] \in (0 :_{R/(x_n,\dots,x_{i+1})R} x_i)_{(*,s+t)} \text{ for } s \geq 0.$$

Assume this is not the case, then either

$$(x^{u^k}/x_i) y^{v^k} y_n^s = x_l x^{u'} y^{v'}$$

for some u', v' and $l \geq i + 1$ which contradicts to $m(u^k) = i$. Or

$$(x^{u^k}/x_i) y^{v^k} y_n^s = x^{u^{k'}} y^{v^{k'}} x^{u'} y^{v'}$$

for $x^{u^{k'}} y^{v^{k'}} \in G(J)$. It follows that $|v'| = s$. Let k_1 be the largest integer l such that $y_n^l | y^{v^{k'}}$. Then

$$(x^{u^k}/x_i) y^{v^k} = ((x^{u^{k'}} y^{v^{k'}} x^{u'}) / y_n^{k_1}) y^{v'} / y_n^{s - k_1} \in J$$

because J is bistable, and this is again a contradiction. Therefore $(*)$ is true and we get $m^i_{s+t} \geq c^i$ for $s \geq 0$. This concludes the proof. $\square$

Remark 5.2. This proposition could also be formulated by changing the roles of $\mathbf{x}$ and $\mathbf{y}$.

Let A be a standard graded K-algebra. For a finitely generated graded A-module M the usual Castelnuovo-Mumford regularity is defined to be

$$\operatorname{reg}^A(M) = \sup\{r \in \mathbb{Z}\colon \beta^A_{i,i+r}(M) \neq 0 \text{ for some integer } i\}.$$

In [7] and [12] it was shown that for a graded ideal $I \subset S_x$ the function $\operatorname{reg}^{S_x}(I^j)$ is a linear function $pj + c$ for $j \gg 0$. In the case that I is generated in one degree we give an upper bound for c and find an integer j_0 for which $\operatorname{reg}^{S_x}(I^j)$ is a linear function for all $j \geq j_0$.

Theorem 5.3. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree $d \in \mathbb{N}$. Let $R(I) = S/J$ for a bigraded ideal J. Then:*

(i) $\operatorname{reg}^{S_x}(I^j) \leq jd + \operatorname{reg}^S_x(R(I))$.
(ii) $\operatorname{reg}^{S_x}(I^j) = jd + c$ *for* $j \geq m_y(\operatorname{bigin}(J))$ *and some constant* $0 \leq c \leq \operatorname{reg}^S_x(R(I))$.

Proof. We choose an almost regular sequence $\tilde{\mathbf{x}} = \tilde{x}_n, \dots, \tilde{x}_1$ for $R(I)$ over S with respect to the x-degree. We have that for all $j \in \mathbb{N}$ the sequence $\tilde{\mathbf{x}}$ is almost regular for I^j over S_x in the sense of [3] because $R(I)_{(*,j)}(-dj) \cong I^j$ as graded S_x-modules and

$$(0 :_{R(I)/(\tilde{x}_n,\dots,\tilde{x}_{i+1})R(I)} \tilde{x}_i)_{(*,j)}(-dj) = (0 :_{I^j/(\tilde{x}_n,\dots,\tilde{x}_{i+1})I^j} \tilde{x}_i).$$

Define m^i_j for $\operatorname{bigin}(J)$ as in 5.1. Since

$$(0 :_{R(I)/(\tilde{x}_n,\dots,\tilde{x}_{i+1})R(I)} \tilde{x}_i) \cong (0 :_{S/((x_n,\dots,x_{i+1})+\operatorname{bigin}(J))} x_i),$$

it follows that

$$jd + m^i_j = r^i_j = \max(\{l\colon (0 :_{I^j/(\tilde{x}_n,\dots,\tilde{x}_{i+1})I^j} \tilde{x}_i)_l \neq 0\} \cup \{0\}).$$

By a characterization of the regularity of graded modules in [3] we have $\operatorname{reg}^{S_x}(I^j) = \max\{jd, r^1_j, \dots, r^n_j\}$. Hence the assertion follows from 4.2, 4.3(ii) and 5.1. $\square$

Similarly as in 5.3 one has:

Theorem 5.4. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree $d \in \mathbb{N}$. Let $S(I) = S/J$ for a bigraded ideal J. Then:*

(i) $\operatorname{reg}^{S_x}(S^j(I)) \leq jd + \operatorname{reg}^S_x(S(I))$.
(ii) $\operatorname{reg}^{S_x}(S^j(I)) = jd + c$ *for* $j \geq m_y(\operatorname{bigin}(J))$ *and some constant* $0 \leq c \leq \operatorname{reg}^S_x(S(I))$.

Blum [4] proved the following with different methods.

Corollary 5.5. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree $d \in \mathbb{N}$.*

(i) *If* $\operatorname{reg}_x(R(I)) = 0$, *then* $\operatorname{reg}^{S_x}(I^j) = jd$ *for* $j \geq 1$.
(ii) *If* $\operatorname{reg}_x(S(I)) = 0$, *then* $\operatorname{reg}^{S_x}(S^j(I)) = jd$ *for* $j \geq 1$.

Proof. This follows from 5.3 and 5.4. $\square$

Next we give a more theoretic bound for the regularity function becoming linear. Consider a bigraded algebra R. Let y be almost regular for all $\mathrm{Tor}_i^S(S/\mathfrak{m}_x, R)$ with respect to the y-degree. Define

$$w(R) = \max\{b\colon (0 :_{\mathrm{Tor}_i^S(S/\mathfrak{m}_x,R)} y)_{(*,b)} \neq 0 \text{ for some } i \in [n]\}.$$

Lemma 5.6. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree $d \in \mathbb{N}$.*

(i) *For $j > w(R(I))$ we have $\mathrm{reg}^{S_x}(I^{j+1}) \geq \mathrm{reg}^{S_x}(I^j) + d$.*

(ii) *For $j > w(S(I))$ we have $\mathrm{reg}^{S_x}(S^{j+1}(I)) \geq \mathrm{reg}^{S_x}(S^j(I)) + d$.*

Proof. We prove the case $R = R(I)$. For $j > w(R)$ one has the exact sequence

$$0 \to \mathrm{Tor}_i^S(S/\mathfrak{m}_x, R)_{(*,j)} \xrightarrow{y} \mathrm{Tor}_i^S(S/\mathfrak{m}_x, R)_{(*,j+1)}.$$

In [7, 3.3] it was shown that

$$\mathrm{Tor}_i^S(S/\mathfrak{m}_x, R)_{(a,j)} \cong \mathrm{Tor}_i^{S_x}(K, I^j)_{a+jd}$$

and this concludes the proof. $\square$

Lemma 5.7. *Let R be a bigraded algebra. Then*

$$H_\bullet(0,m)_{(*,j)} = 0 \textit{ for } j > \mathrm{reg}_y(R) + m.$$

Proof. We know that

$$H_\bullet(0,m) \cong \mathrm{Tor}_\bullet^S(S/\mathfrak{m}_y, R) \cong H_\bullet(S/\mathfrak{m}_y \otimes_S F_\bullet)$$

where $F_\bullet$ is the minimal bigraded free resolution of R over S. Let

$$F_i = \bigoplus S(-a,-b)^{\beta^S_{i,(a,b)}(R)}.$$

Then by the definition of the y-regularity we have $b \leq \mathrm{reg}_y(R) + m$ for all $\beta^S_{i,(a,b)}(R) \neq 0$. Thus $(S/(\mathbf{y}) \otimes_S F_i)_{(*,j)} = 0$ for $j > \mathrm{reg}_y(R) + m$. The assertion follows. $\square$

We get the following exact sequences.

Corollary 5.8. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree $d \in \mathbb{N}$.*

(i) *For $j > \mathrm{reg}_y(R(I)) + m$ we have the exact sequence*

$$0 \to I^{j-m}(-md) \to \bigoplus_m I^{j-m+1}(-(m-1)d) \to \dots \to \bigoplus_m I^{j-1}(-d) \to I^j \to 0.$$

(ii) *For $j > \mathrm{reg}_y(S(I)) + m$ we have the exact sequence*

$$0 \to S^{j-m}(I)(-md) \to \bigoplus_m S^{j-m+1}(I)(-(m-1)d) \to$$

$$\dots \to \bigoplus_m S^{j-1}(I)(-d) \to S^j(I) \to 0.$$

Proof. This statement follows from 5.7 and the fact that $R(I)_{(*,j)}(-jd) \cong I^j$ or $S(I)_{(*,j)}(-jd) \cong S^j(I)$ respectively. $\square$

Corollary 5.9. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree $d \in \mathbb{N}$. Then:*

(i) *For $j \geq \max\{\operatorname{reg}_y(R(I)) + m, w(R(I)) + m\}$ we have*
$$\operatorname{reg}^{S_x}(I^{j+1}) = d + \operatorname{reg}^{S_x}(I^j).$$
(ii) *For $j \geq \max\{\operatorname{reg}_y(S(I)) + m, w(S(I)) + m\}$ we have*
$$\operatorname{reg}^{S_x}(S^{j+1}(I)) = d + \operatorname{reg}^{S_x}(S^j(I)).$$

Proof. We prove the corollary for $R(I)$. By 5.8 and by standard arguments (see 6.1 for the bigraded case) we get that for $j \geq \operatorname{reg}_y(R(I)) + m$
$$\operatorname{reg}^{S_x}(I^{j+1}) \leq \max\{\operatorname{reg}^{S_x}(I^{j+1-i}) + id - i + 1 \colon i \in [m]\}.$$
Since $j + 1 - i > w(R(I))$, it follows from 5.6 that
$$\operatorname{reg}^{S_x}(I^{j+1-i}) \leq \operatorname{reg}^{S_x}(I^{j+1-i+1}) - d \leq \ldots \leq \operatorname{reg}^{S_x}(I^{j+1}) - id.$$
Hence $\operatorname{reg}^{S_x}(I^{j+1}) = \operatorname{reg}^{S_x}(I^j) + d$. ∎

We now consider a special case where $\operatorname{reg}^{S_x}(I^j)$ can be computed precisely.

Proposition 5.10. *Let $R = S/J$ be a bigraded algebra which is a complete intersection. Let $\{z_1, \dots, z_t\}$ be a homogeneous minimal system of generators of J which is a regular sequence. Assume that $\deg_x(z_t) \geq \ldots \geq \deg_x(z_1) > 0$ and $\deg_y(z_k) = 1$ for all $k \in [t]$. Then for all $j \geq t$*
$$\operatorname{reg}^{S_x}(R_{(*,j+1)}) = \operatorname{reg}^{S_x}(R_{(*,j)}).$$
If in addition $\deg_x(z_k) = 1$ for all $k \in [t]$, then for $j \geq 1$
$$\operatorname{reg}^{S_x}(R_{(*,j)}) = 0.$$

Proof. The Koszul $K_\bullet(\mathbf{z})$ complex with respect to $\{z_1, \dots, z_t\}$ provides a minimal bigraded free resolution of R because these elements form a regular sequence. Observe that $(*, j)$ is an exact functor on complexes of bigraded modules. Note that $K_\bullet(\mathbf{z})_{(*,j)}$ is a complex of free S_x-modules because
$$K_i(\mathbf{z}) \cong \bigoplus_{\{j_1,\dots,j_i\} \subseteq [t]} S(-\deg(z_{j_1}) - \ldots - \deg(z_{j_i})),$$
and
$$S(-a,-b)_{(*,j)} \cong \bigoplus_{|v|=j-b} S_x(-a)y^v \text{ as graded } S_x\text{-modules.}$$
Furthermore $K_\bullet(\mathbf{z})_{(*,j)}$ is minimal by the additional assumption $\deg_x(z_k) > 0$. We have for $j \geq t$
$$\operatorname{reg}^{S_x}(R_{(*,j)}) = \max\{\deg_x(z_t) + \ldots + \deg_x(z_{t-i+1}) - i \colon i \in [t]\}$$
and this is independent of j. If in addition $\deg_x(z_k) = 1$ for all k, then we obtain
$$\operatorname{reg}^{S_x}(R_{(*,j)}) = 0 \text{ for } j \geq 1.$$
∎

Recall that a graded ideal I is said to be of linear type, if $R(I) = S(I)$. For example ideals generated by d-sequences are of linear type. Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal, which is Cohen-Macaulay of codim 2. By the Hilbert-Burch theorem S_x/I has a minimal graded free resolution

$$0 \to \bigoplus_{i=1}^{m-1} S_x(-b_i) \xrightarrow{B} \bigoplus_{i=1}^{m} S_x(-a_i) \to S_x \to S_x/I \to 0$$

where $B = (b_{ij})$ is a $m \times m-1$-matrix with $b_{ij} \in \mathfrak{m}$ and we may assume that the ideal I is generated by the maximal minors of B. The matrix B is said to be the Hilbert-Burch matrix of I. If I is generated in degree d, then $S(I) = S/J$ where J is the bigraded ideal $(\sum_{i=1}^m b_{ij} y_i \colon j = 1, \dots, m-1)$.

Corollary 5.11. *Let $I = (f_1, \dots, f_m) \subset S_x$ be a graded ideal generated in degree $d \in \mathbb{N}$, which is Cohen-Macaulay of codim 2 with $m \times m-1$ Hilbert-Burch matrix $B = (b_{ij})$ and of linear type. Then for $j \geq m-1$*

$$\operatorname{reg}^{S_x}(I^{j+1}) = \operatorname{reg}^{S_x}(I^j) + d.$$

If additionally $\deg_x(b_{ij}) = 1$ for $b_{ij} \neq 0$, then the equality holds for $j \geq 1$.

Proof. Since I is of linear type, we have $R(I) = S(I) = S/J$ with the ideal $J = (\sum_{i=1}^m b_{ij} y_i \colon j = 1, \dots, m-1)$. One knows that (Krull-) $\dim(R(I)) = n+1$. Since J is defined by $m-1$ equations, we conclude that $R(I)$ is a complete intersection. Now apply 5.10. $\square$

6. Bigraded Veronese algebras

Let R be a bigraded algebra and fix $\tilde{\Delta} = (s,t) \in \mathbb{N}^2$ with $(s,t) \neq (0,0)$. We call

$$R_{\tilde{\Delta}} = \bigoplus_{(a,b) \in \mathbb{N}^2} R_{(as,bt)}$$

the *bigraded Veronese algebra of R* with respect to $\tilde{\Delta}$ (see for example [9] for the graded case and [6] for similar constructions in the bigraded case). Note that $R_{\tilde{\Delta}}$ is again a bigraded algebra. We want to relate $\operatorname{reg}_x^{S_{\tilde{\Delta}}}(R_{\tilde{\Delta}})$ and $\operatorname{reg}_y^{S_{\tilde{\Delta}}}(R_{\tilde{\Delta}})$ to $\operatorname{reg}_x^S(R)$ and $\operatorname{reg}_y^S(R)$. We follow the way presented in [6] for the case of diagonals.

Lemma 6.1. *Let R be a bigraded algebra and*

$$0 \to M_r \to \dots \to M_0 \to N \to 0$$

be an exact complex of finitely generated bigraded R-modules. Then

$$\operatorname{reg}_x^R(N) \leq \sup\{\operatorname{reg}_x^R(M_k) - k \colon 0 \leq k \leq r\}$$

and

$$\operatorname{reg}_y^R(N) \leq \sup\{\operatorname{reg}_y^R(M_k) - k \colon 0 \leq k \leq r\}.$$

Proof. We prove by induction on $r \in \mathbb{N}$ the inequality above for $\operatorname{reg}_x^R(N)$. The case $r = 0$ is trivial. Now let $r > 0$, and consider

$$0 \to N' \to M_0 \to N \to 0$$

where N' is the kernel of $M_0 \to N$. Then for every integer a we have the exact sequence

$$\ldots \to \operatorname{Tor}_i^R(M_0, K)_{(a+i,*)} \to \operatorname{Tor}_i^R(N, K)_{(a+i,*)} \to \operatorname{Tor}_{i-1}^R(N', K)_{(a+1+i-1,*)} \to \ldots$$

We get

$$\operatorname{reg}_x^R(N) \leq \sup\{\operatorname{reg}_x^R(M_0),\ \operatorname{reg}_x^R(N') - 1\} \leq \sup\{\operatorname{reg}_x^R(M_k) - k \colon 0 \leq k \leq r\}$$

where the last inequality follows from the induction hypothesis. Analogously we obtain the inequality for $\operatorname{reg}_y^R(N)$. □

Lemma 6.2. *Let A and B be graded K-algebras, M be a finitely generated graded A-module and N be a finitely generated graded B-module. Then $M \otimes_K N$ is a finitely generated bigraded $A \otimes_K B$-module with*

$$\operatorname{reg}_x^{A\otimes_K B}(M \otimes_K N) = \operatorname{reg}^A(M) \text{ and } \operatorname{reg}_y^{A\otimes_K B}(M \otimes_K N) = \operatorname{reg}^B(N).$$

Proof. Let $F_\bullet$ be the minimal graded free resolution of M over A and $G_\bullet$ be the minimal graded free resolution of N over B. Then $H_\bullet = F_\bullet \otimes_K G_\bullet$ is the minimal bigraded free resolution of $M \otimes_K N$ over $A \otimes_K B$ with $H_i = \bigoplus_{k+l=i} F_k \otimes G_l$. Since $A(-a) \otimes_K B(-b) = (A \otimes_K B)(-a, -b)$, the assertion follows. □

Theorem 6.3. *Let R be a bigraded algebra, $\tilde{\Delta} = (s,t) \in \mathbb{N}^2$ with $(s,t) \neq (0,0)$. Then*

$$\operatorname{reg}_x^{S_{\tilde{\Delta}}}(R_{\tilde{\Delta}}) \leq \max\{c \colon c = \lceil a/s \rceil - i, \beta_{i,(a,b)}^S(R) \neq 0 \text{ for some } i, b \in \mathbb{N}\}$$

and

$$\operatorname{reg}_y^{S_{\tilde{\Delta}}}(R_{\tilde{\Delta}}) \leq \max\{c \colon c = \lceil b/t \rceil - i, \beta_{i,(a,b)}^S(R) \neq 0 \text{ for some } i, a \in \mathbb{N}\}.$$

Proof. By symmetry it suffices to show the inequality for $\operatorname{reg}_x^{S_{\tilde{\Delta}}}(R_{\tilde{\Delta}})$. Let

$$0 \to F_r \to \ldots \to F_0 \to R \to 0$$

be the minimal bigraded free resolution of R over S. Since $()_{\tilde{\Delta}}$ is an exact functor, we obtain the exact complex of finitely generated $S_{\tilde{\Delta}}$-modules

$$0 \to (F_r)_{\tilde{\Delta}} \to \ldots \to (F_0)_{\tilde{\Delta}} \to R_{\tilde{\Delta}} \to 0.$$

By 6.1 we have

$$\operatorname{reg}_x^{S_{\tilde{\Delta}}}(R_{\tilde{\Delta}}) \leq \max\{\operatorname{reg}_x^{S_{\tilde{\Delta}}}((F_i)_{\tilde{\Delta}}) - i\}.$$

Since

$$F_i = \bigoplus_{(a,b)\in\mathbb{N}^2} S(-a,-b)^{\beta_{i,(a,b)}^S(R)},$$

one has

$$\operatorname{reg}_x^{S_{\tilde{\Delta}}}((F_i)_{\tilde{\Delta}}) = \max\{\operatorname{reg}_x^{S_{\tilde{\Delta}}}(S(-a,-b)_{\tilde{\Delta}}) \colon \beta_{i,(a,b)}^S(R) \neq 0\}.$$

We have to compute $\operatorname{reg}_x^{S_{\tilde\Delta}}(S(-a,-b)_{\tilde\Delta})$. Let $M_0,\dots,M_{s-1}$ be the relative Veronese modules of S_x and $N_0,\dots,N_{t-1}$ be the relative Veronese modules of S_y. That is $M_j=\bigoplus_{k\in\mathbb{N}}(S_x)_{ks+j}$ for $j=0,\dots,s-1$ and $N_j=\bigoplus_{k\in\mathbb{N}}(S_y)_{kt+j}$ for $j=0,\dots,t-1$. Then

$$S(-a,-b)_{\tilde\Delta}=\bigoplus_{(k,l)\in\mathbb{N}^2}(S_x)_{ks-a}\otimes_K(S_y)_{lt-b}=M_i(-\lceil a/s\rceil)\otimes_K N_j(-\lceil b/t\rceil)$$

where $i\equiv -a \mod s$ for $0\le i\le s-1$ and $j\equiv -b \mod t$ for $0\le j\le t-1$.

By [1] the relative Veronese modules over a polynomial ring have a linear resolution over the Veronese algebra. Hence 6.2 yields $\operatorname{reg}_x^{S_{\tilde\Delta}}(S(-a,-b)_{\tilde\Delta})=\lceil a/s\rceil$. This concludes the proof. □

Corollary 6.4. *Let R be a bigraded algebra.*

(i) *For $s\gg 0, t\in\mathbb{N}$ and $\tilde\Delta=(s,t)$ one has* $\operatorname{reg}_x^{S_{\tilde\Delta}}(R_{\tilde\Delta})=0$.
(ii) *For $t\gg 0, s\in\mathbb{N}$ and $\tilde\Delta=(s,t)$ one has* $\operatorname{reg}_y^{S_{\tilde\Delta}}(R_{\tilde\Delta})=0$.

References

[1] A. Aramova, S. Barcanescu and J. Herzog, On the rate of relative Veronese submodules, *Rev. Roumaine Math. Pures Appl.* **40** (1995), no. 3-4, 243-251.
[2] A. Aramova, K. Crona and E. De Negri, Bigeneric initial ideals, diagonal subalgebras and bigraded Hilbert functions, *J. Pure Appl. Algebra* **150** (2000), no. 3, 215-235.
[3] A. Aramova and J. Herzog, Almost regular sequences and Betti numbers, *Amer. J. Math.* **122** (2000), no. 4, 689-719.
[4] S. Blum, personal communication (2000).
[5] W. Bruns and J. Herzog, Cohen-Macaulay rings, revised edition, *Cambridge Studies in Advanced Mathematics* **39**, Cambridge Univ. Press, Cambridge, 1998.
[6] A. Conca, J. Herzog, N. V. Trung and G. Valla, Diagonal subalgebras of bigraded algebras and embeddings of blow-ups of projective spaces, *Amer. J. Math.* **119** (1997), no. 4, 859-901.
[7] S. D. Cutkosky, J. Herzog and N. V. Trung, Asymptotic behaviour of the Castelnuovo-Mumford regularity, *Compositio Math.* **118** (1999), no. 3, 243-261.
[8] D. Eisenbud, Commutative algebra with a view toward algebraic geometry, *Graduate Texts in Mathematics* **150**, Springer-Verlag, New York, 1995.
[9] D. Eisenbud, A. Reeves and B. Totaro, Initial ideals, Veronese subrings, and rates of algebras, *Adv. Math.* **109** (1994), no. 2, 168-187.
[10] J. Herzog, G. Restuccia and Z. Tang, s-sequences and symmetric algebras, preprint (2000).
[11] C. Huneke, The theory of d-sequences and powers of ideals, *Adv. in Math.* **46** (1982), no. 3, 249-279.
[12] V. Kodiyalam, Asymptotic behaviour of Castelnuovo-Mumford regularity, *Proc. Amer. Math. Soc.* **128** (2000), no. 2, 407-411.
[13] N. V. Trung, The Castelnuovo regularity of the Rees algebra and the associated graded ring, *Trans. Amer. Math. Soc.* **350** (1998), no. 7, 2813-2832.

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